Elektriciteit en magnetisme 2
Instructor: A.M. van den Berg

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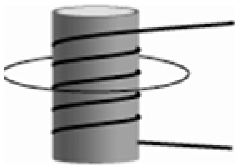
(1) (Total 20 marks)
A solenoid is made from a long wire, twisted in 5 turns around a tube. A current I_s flows through this wire. The length of the solenoid is 38 cm and the diameter of the solenoid is 20 cm.

(a) (10 marks) Calculate on the axis of the solenoid the magnetic field under the assumption that the solenoid has infinite length and that the value of the current is given as $I_s = 80$ mA.

(b) (10 marks) One puts around the solenoid a current loop. The plane of the current loop is perpendicular to the axis of the solenoid. The current is a single loop with a diameter of 25 cm and it has a resistance $R = 12\ \Omega$. One changes the current I_s from 80 mA to 300 mA. Calculate the induced current in the current loop.



SOLUTION

(a) The magnetic field of an infinity long solenoid is homogeneous inside the solenoid and equal to 0 outside the solenoid. The field strength inside (on thus on the axis) is given as:

$$\vec{B} = \frac{\mu_0\ N\ I_s}{\ell}\hat{z}$$

where N is the number of turns (5) and ℓ is the length of the solenoid (0.38 m). For a current $I_s = 80$ mA, we can calculate:

$$B = \frac{4\pi \times 10^{-7} \times 5 \times 80 \times 10^{-3}}{0.38} = 1.32 \times 10^{-6}\ \mathrm{T}$$

(b) The **emf** is given by Faraday's law:

$$\mathrm{emf} = -\frac{d}{dt}\int_S \vec{B} \cdot \hat{n}\ da$$

This is the equation for the change in the magnetic flux. Keep in mind that the flux is contained to the diameter of the solenoid, so the integration over the surface is limited to $d = 20$ cm or $r_s = 10$ cm;

$$\mathrm{emf} = -\pi r_s^2 \frac{dB}{dt} = -\pi r_s^2 \frac{\mu_0\ N}{\ell}\frac{dI_s}{dt}$$

The question was lacking the value of dt (don't worry I was flexible in my grading here), but if we assume that $dt = 2$ s and given that $dI = 80 - 300$ mA = -0.220 A, we can calculate the emf as:

$$\text{emf} = \pi \times 0.1^2 \, \frac{4\,\pi \times 10^{-7} \times 5}{0.38} \frac{(-0.22)}{2} = -2.6 \times 10^{-7}\text{V}$$

With the given value of $R = 12\ \Omega$, we find that $I_{loop} = \text{emf}/R = 2.2 \times 10^{-7}$ A

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(2) (Total 20 marks)
The electric field of an electromagnetic wave is given as:

$$\vec{E} = E_0 \ \sin(\pi \ z/z_0) \ \cos(kx - \omega t)\hat{y}$$

(a) (5 marks) What is the polarization direction of the wave?

The generic wave equation for electromagnetic waves is given as:

$$\nabla^2 \vec{E} = \epsilon_0 \mu_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

(b) (15 marks) Use this wave equation to derive an expression for the wave number k of the electromagnetic wave.
(c) (5 marks) Give an expression for the velocity of propagation of the electromagnetic wave.

SOLUTION
(a) By definition, the polarization direction is given by the component of the electric field vector; thus the polarization is in the y-direction
(b) The wave equation is given as

$$\nabla^2 \vec{E} = \epsilon_0 \mu_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

where the ∇^2 operator (Laplacian) is $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$. The wave depends only on z, x, and t; and for each of these variables we have to take the second derivative for a cos or a sin; this yields:

$$\left[-k^2 - \frac{\pi^2}{z_0^2} + \epsilon_0 \mu_0 \omega^2\right] E_0 \sin(\pi \ z/z_0) \ \cos(kx - \omega t) = 0$$

This must be true for all values of z, x, and t, therefore the expression in brackets must be set to zero.

$$-k^2 - \frac{\pi^2}{z_0^2} + \epsilon_0 \mu_0 \omega^2 = 0$$

This we can solve for k as:

$$k = \pm \sqrt{\epsilon_0 \mu_0 \omega^2 - \left(\frac{\pi}{z_0}\right)^2}$$

(c) The propagation velocity is $v = \omega/k$ (Note the group velocity is $\frac{d\omega}{dk}$ and thus:

$$v = \frac{\omega}{k} = \pm \frac{\omega}{\sqrt{\epsilon_0 \mu_0 \omega^2 - \left(\frac{\pi}{z_0}\right)^2}}$$

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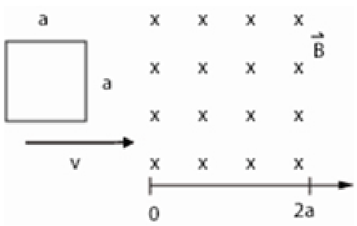
(3) (Total 20 marks)
A square current loop moves at a constant velocity v through a magnetic field in the positive x-direction. The length of each side of the current loop is a cm. For the area $0 < x < 2a$ and $0 < y < 2a$ the magnetic field strength has a value $B = B_0$ and it is perpendicular to the plane of the current loop (i.e. in the z-direction). Outside the area $0 < x < 2a$ and $0 < y < 2a$ the strength of the magnetic field is $B = 0$ T.

(a) (10 marks) Calculate the maximum value of the electromotive force, induced in the square current loop.

(b) (10 marks) Plot a graph of the electromotive force induced in the square current loop as a function of the x-coordinate.



SOLUTION

(a) Faraday's law tells us that the emf is given as the change in the flux:

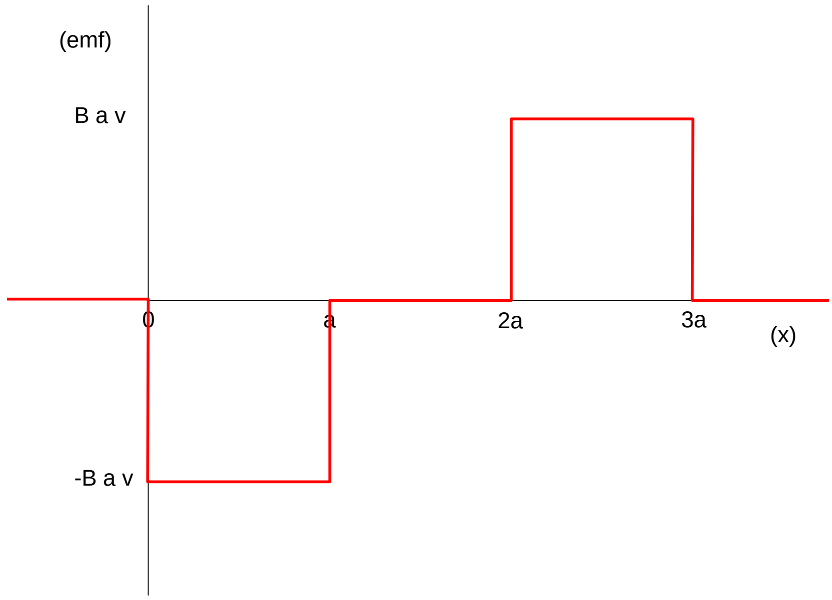
$$\text{emf} = -\frac{d\Phi_B}{dt} = -\frac{d}{dt}\int_S \vec{B} \circ \hat{n}\, da$$

where $\vec{B}$ and $\hat{n}$ are parallel (we can take the value of B here) and da integrates over the square of the loop, but only for that part where the loop is inside the magnetic field.

For $x < 0$ $\quad \Phi = 0$
For $0 < x < a$ $\quad \Phi = B\ a\ x$
For $a < x < 2a$ $\quad \Phi = B\ a^2$
For $2a < x < 3a$ $\quad \Phi = B\ a\ (3a - x)$
For $3a < x$ $\quad \Phi = 0$

We take the derivative with respect to t in all cases ($dx/dt = v$) and find the maximum value for the emf as:

$$\text{emf}_{max} = B\ a\ v$$



(b)

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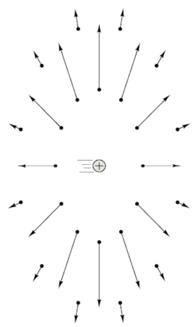
(4) (Total 20 marks)
A charge q is at rest in the inertial frame S_0 and it is located at the origin. Another inertial frame S moves with a constant velocity v in the positive x-direction relative to the frame S_0. The electric field of this charge in frame S is given as:

$$\vec{E} = \frac{q}{4\pi\epsilon_0} \frac{1 - v^2/c^2}{\left[1 - (v^2/c^2)\sin^2\theta\right]^{3/2}} \frac{\vec{R}}{R^3}$$

where θ is the angle between the vector $\vec{R}$ and the velocity v.



(a) (5 marks) Give the expressions of the electric field and the magnetic field in frame S_0.
(b) (5 marks) Calculate in inertial frame S_0 the Poynting vector.
(c) (5 marks) Give an expression for the magnetic field in inertial frame S.
(d) (5 marks) Calculate in inertial frame S the Poynting vector.

SOLUTION

(a) In the frame S_0 the charge q doesn't move, so $\beta = 0$ and you find the law of Coulomb for the electric field. The charge is not moving, so the magnetic field is zero;

$$E = \frac{q}{4\pi\epsilon_0}\frac{\vec{R}}{R^3}$$

and

$$B = 0$$

(b) The Poynting vector is given as:

$$\vec{P} = \frac{1}{\mu_0}\left(\vec{E} \times \vec{B}\right)$$

Because $\vec{B} = 0$, the Poynting vector is $\vec{P} = 0$

(c) If there is an inertial frame where $\vec{B} = 0$ (as it is in the present case), we can calculate the magnetic field in another inertial frame (S) from the electric field vector in that other frame (S) as:

$$\vec{B} = \frac{1}{c^2}\left(\vec{v} \times \vec{E}\right)$$

where v is the velocity between the frames S and S_0. The electric field in frame S is given as:

$$\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{1 - v^2/c^2}{\left[1 - (v^2/c^2)\sin^2\theta\right]^{3/2}}\frac{\vec{R}}{R^3}$$

Thus the magnetic field vector is then:

$$B = \frac{1}{c^2}\frac{q}{4\pi\epsilon_0}\frac{1 - v^2/c^2}{\left[1 - (v^2/c^2)\sin^2\theta\right]^{3/2}}\frac{\vec{v} \times \vec{R}}{R^3}$$

(d) The Poynting vector in the frame S can be evaluated by looking at the vectorial product $\vec{E} \times \vec{B}$, where we note that this product in the present case involves the following product:

$$\frac{\vec{R}}{R^3} \times \frac{\vec{v} \times \vec{R}}{R^3}$$

Following the rules of out products, this multiplication results in the value 0. So als in this case $\vec{P} = 0$.

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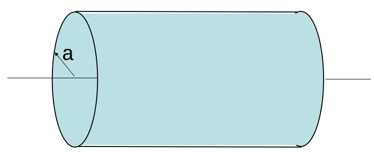
(5) (Total 20 marks)
A coaxial cable consists of a core and a mantle. The radius of the mantle has a value a. A current I flows through the core of the conductor and back through the mantle. The current changes in time as: $I = I_0 \cos(\omega t)$.



(a) (10 marks) Give an expression of the magnetic field strength for $r > a$ and for $0 < r < a$.

(b) (10 marks) Calculate the induced electric field strength for $r > a$ and for $0 < r < a$.

SOLUTION

(a) Here I will assume that the forth and back running currents are in phase and that the cross sections of the core and of the mantle are infinitely small; in other words the currents are limited to the center $s = 0$ (forth) and $s = a$ (back). Again, I'll be flexible in grading. Use the law of Ampère in both ranges $s < a$ and $s > a$.
For $0 < s < a$, we find

$$\oint \vec{B} \circ d\vec{\ell} = \mu_0 I_{enclosed}$$

Thus in this case:

$$B = \frac{\mu_0}{2\pi\ s} I_0 \cos(\omega t)$$

Because of symmetry reasons, the magnetic field field vector is in the ϕ direction.

$$\vec{B} = \frac{\mu_0}{2\pi\ s} I_0 \cos(\omega t)\hat{\phi}$$

For $s > a$, the enclosed current is zero, thus here we find $B = 0$.

(b) The magnetic field changes only in the region $0 < s < a$, for $s > a$ there will be no induced electric field. To calculate the induced electric field for $0 < s < a$, we use the equation: $\vec{\nabla} \times \vec{E} = -\partial\vec{B}/\partial t$, where we note that only the ϕ direction of the B-field gives a contribution. Furthermore, because of symmetry, the E-field will be in the z-direction. Using the curl operator in cylindrical coordinates we find:

$$\left(\vec{\nabla} \times \vec{E}\right)_\phi = \left[\frac{\partial E_s}{\partial z} - \frac{\partial E_z}{\partial s}\right]_\phi$$

Thus we can write this as:

$$\frac{\partial E_z}{\partial s} = -\frac{\partial B_\phi}{\partial t} = -\frac{\mu_0 I_0}{2\pi s}\frac{\partial}{\partial t}\cos(\omega t) = \frac{\omega\mu_0 I_0}{2\pi s}\sin(\omega t)$$

We now integrate thus over s, where the only relevant part in the integral is $\frac{1}{s}$; the integral of this function is $\ln(s)$. Therefore, E_z is given as:

$$E_z = \frac{\omega\mu_0 I_0}{2\pi}\sin(\omega t)\left[A - \ln(s)\right]$$

Here A denotes an integration constant, which we can calculate easily, because we note that for $s = a$ the electric field must go to zero (remind the solution for $s > a$, we found that $E = 0$). Thus the value of $A = \ln(a)$, and

$$E_z = \frac{\omega\mu_0 I_0}{2\pi}\sin(\omega t)\left[\ln(a) - \ln(s)\right]$$

Alternatively, you can also calculate the line integral over the E-field in a rectangle with length L. Here the rectangle has one leg with length L parallel to the axis inside the radius a, the "return leg" is at radius $s > a$ and the two joints are in between. Note that these two joints and the path integral outside radius $s > a$ don't contribute, because for the "joints" the field is perpendicular to the path, and for the "return leg", the field is zero.

$$\oint \vec{E} \circ d\vec{\ell} = E\ L = -\frac{d}{dt}\Phi_B = -\frac{d}{dt}\int \vec{B} \circ \hat{n}\ da$$

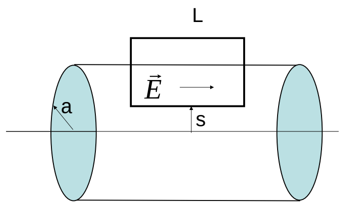
where da is the area of the loop (it is not a circle, because the loop runs parallel with the z-axis). Therefore,

$$E\ L = -\frac{d}{dt}\int_s^a \int_0^L \frac{\mu_0 I_0}{2\pi s'}\cos(\omega t) d\ell\ ds'$$

The integration over $d\ell$ and s' and the derivative with respect to t can be interchanged and the results is straight forward:

$$E = \frac{\mu_0 I_0}{2\pi}\omega \ln\frac{a}{r}\sin(\omega t)$$

And we remember that the E-field is in the z-direction only.



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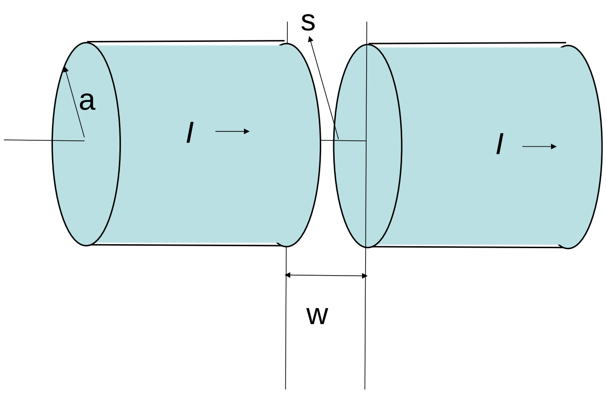
(6) (Total 20 marks)
A fat wire, with radius a, carries a constant current I, uniformly distributed over its cross section. A narrow gap in the wire, of width $w \ll a$, forms a parallel-plate capacitor, as shown in the figure. Find the magnetic field in the gap at a distance $s < a$ from the axis.



SOLUTION

The current I is not running in between the plates, but charges are deposited at the interface between the plates. Under the assumption, that the plates of the capacitor are very close to each other ($w \ll a$), we can write the electric field between the two plates as:

$$E = \frac{1}{\epsilon_0}\sigma = \frac{1}{\epsilon_0}\frac{Q}{A}$$

where A is the area of the plate, in this case $A = \pi a^2$. In between the plates there is a changing electric field, because of the increasing number of charges on the plates;

$$\frac{\partial E}{\partial t} = \frac{1}{\epsilon_0\ A}\frac{dQ}{dt} = \frac{1}{\epsilon_0\ A}I = \frac{1}{\epsilon_0\ \pi a^2}I$$

This changing electric field is directly related to the displacement current given in Maxwell's equation:

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}_{free} + \mu_0 \epsilon_0 \frac{\partial E}{\partial t}$$

where the free current is zero (no free charges flow between the plates) and the second term on the RHS we have derived above (directly related to the displacement current). We can calculate the magnetic field with an Amperian loop with radius s centered at the axis. Note, that the area of the loop defines the ENCLOSED current.

$$I_{enclosed} = I\ \frac{s^2}{a^2}$$

Therefore we find:

$$B\ 2\ \pi\ s = \mu_0 I_{enclosed} = \mu_0\ I\ \frac{s^2}{a^2}$$

Solving this for B gives us:

$$B = \frac{\mu_0\ I\ s}{2\ \pi\ a^2}$$